AIAA JOURNAL
Vol. 37, No. 11, November 1999

Extraction of Substructural Flexibility from Global Frequencies and Mode Shapes

K. F. Alvin*
Sandia National Laboratories, Albuquerque, New Mexico 87185-0439
and
K. C. Park†
University of Colorado, Boulder, Colorado 80309

A computational procedure for extracting substructure-by-substructure flexibility properties from global modal parameters is presented. The present procedure consists of two key features: an element-based direct flexibility method, which uniquely determines the global flexibility without resorting to case-dependent redundancy selections, and the projection of kinematically inadmissible modes that are contained in the iterated substructural matrices. The direct flexibility method is used as the basis of an inverse problem, whose goal is to determine substructural flexibilities given the global flexibility, geometrically determined substructural rigid-body modes, and the local-to-global assembly operators. The resulting procedure, given accurate global flexibility, extracts the exact element-by-element substructural flexibilities for determinate structures. For indeterminate structures, the accuracy depends on the iteration tolerance limits. The procedure is illustrated using both simple and complex numerical examples and appears to be effective for structural applications such as damage localization and finite element model reconciliation.

I. Introduction

INVERSE problems in linear structural dynamics, and in particular inverse structural modeling, have been the subject of intense research interest during the past 10 years. Inverse structural modeling encompasses structural identification,[1,2] finite element model updating,[3,4] and damage detection.[5–11] Advances made in these three categories have greatly benefited computational model validation, active structural vibration control strategies, design improvement of mechanical systems subject to dynamic operating conditions, and damage assessment for aging structural systems such as aircraft and surface ships, offshore platforms, bridges, and high-rise buildings.

The identified structural model parameters used in such endeavors consist of structural vibration mode shapes, frequencies, and damping rates. These modal quantities are global properties by nature. Model changes, however, occur most often because of changes in the local elemental or substructural conditions, as is often the case when a substructure significantly loses stiffness due to damage. Therefore, studies have been focused on how to accentuate the sensitivity of the global properties to capture the changes in local structural properties. Several applications of these techniques have demonstrated that damage can be detected provided the local changes bring about a noticeable change in the global vibration characteristics.

There are several important situations wherein a sharper estimate of localized changes in stiffness and/or damping is demanded. These include structural integrity of joints in high-rise buildings subjected to strong wind and earthquakes, offshore oil platforms where catastrophic failure can emanate from localized damage, loss of redundancies of trusslike structures, and aircraft/engine crack propagation. The objective here is to offer a method for extracting localized flexibility from estimates of the global flexibility, obtained either indirectly from the summation of modal and residual flexibilities (which are themselves obtained by extracting the frequencies, mass-normalized mode shapes, and residuals from modal data) or directly by processing measured vibration signals.

The present procedure is related to two recent trends in inverse structural modeling: flexibility-based methods and disassembly of structural matrices. Flexibility-based methods involve the use of flexibility matrices as a basis for parameter estimation and test–analysis model reconciliation. A key motivation for using flexibility methods has been to effectively condense the frequency and mode shape information from a large number of modes into a reduced set of structural model parameters that have a clear mechanical interpretation. This condensation is useful both for identification of reduced structural matrices and for reconciliation of complex models, where attempts to reconcile large numbers of modes often leads to ambiguous or contradictory parameter estimates. Robinson et al.,[12] used flexibilities derived from modal test parameters to perform localization of hidden damage in aircraft structures, whereas Denoyer and Peterson[13,14] developed finite element model updating procedures based upon flexibility matrices.

The other recent trend in inverse structural modeling, analytical disassembly, refers to algorithms that attempt to identify substructural matrices that, when assembled through assumed compatibility and equilibrium conditions, yield a known or identified global matrix. Doebling et al.[15] and Doebling[16] developed disassembly procedures for both stiffness and flexibility matrices that involve the decomposition of the elemental matrices into elemental eigenvectors, which are dependent only on known quantities (geometry and assumed shape functions), and elemental eigenvalues, which are directly a function of the parameters being identified. Hemez[17] used a similar disassembly decomposition to efficiently compute sensitivities for frequency-response-function-based model updating. Finally, Gordis[18] examined the stiffness disassembly problem and concluded that disassembly was only possible for determinate beam-like structures. This conclusion is incorrect, however, because it fails to account for constraints governing the disassembly, such as conservation of elemental rigid-body modes and the required block-diagonal character of the resultant matrix containing the element-by-element stiffness matrices.

From a mathematical viewpoint, the present procedure involves two related tasks: assembly of the global flexibility from substructural or elemental flexibilities and disassembly of the global flexibility matrix into substructural flexibility matrices. For the assembly of the global flexibility matrix, classical force methods exist (see Refs. 19 and 20, in particular). In recent years, Lagrange multiplier

Received 18 December 1996; revision received 20 January 1999; accepted for publication 5 April 1999.

*Senior Member of Technical Staff, P.O. Box 5800, Structural Dynamics and Vibration Control Department. Senior Member AIAA

†Professor, Campus Box 429, Department of Aerospace Engineering Sciences and Center for Aerospace Structures. Associate Fellow AIAA.

methods[21–23] have been proposed for the solution of large-scale structures on parallel computers. The present algorithm is based on a direct flexibility method[22,23] that not only partitions the global flexibility matrix into substructural flexibility matrices but also effectively assembles global flexibility from substructural flexibility. This flexibility assembly provides the basis of an inverse problem. The inverse problem is a form of disassembly that uses the global flexibility matrix to arrive at estimates of localized substructural or element flexibility. The elements or substructures are defined herein with respect to a set of measured degrees of freedom (DOFs). Thus, no mode shape expansions are utilized, and the extracted local flexibilities are equivalent to analytic model matrices that are reduced or condensed to the same DOFs. The disassembly is different than that of Refs. 15 and 17 because it does not assume the form of the elemental eigenvectors. The key constraints that operate on the inverse algorithm are the preservation of substructural rigid-body modes and the block-diagonal form of the estimated substructure-by-substructure flexibility matrix. The inverse formulation leads to a complex nonlinear matrix equation, which is solved in a iterative fashion. The aforementioned constraints are imposed on the estimated result at each iteration.

The remainder of the paper is organized as follows. In Sec. II, the classical force method for assembly of global flexibility is reviewed. We conclude that such a nonsystematic approach, which was abandoned for the most part in favor of the systematic displacement method for structural analysis, is not an appropriate basis for the inverse problem. In Sec. III, the direct and systematic flexibility method is developed, which provides the basis of the present procedure. In Sec. IV, the inverse problem is derived mathematically, and solution methods are developed. Then, in Secs. V and VI, numerical examples are used to illustrate the procedure on both simple and complex problems. Finally, conclusions are offered in Sec. VII.

II. Determination of Substructural Flexibility via Classical Force Method

A typical structural identification procedure provides the structural mode shapes $\Phi(n, m)$ and modal frequencies $\Omega(m, m)$, where m is the number of identified modes and n is the number of measured DOFs. These data may be used for improvement and validation of an analytical mathematical model, i.e., finite element model, of the structure, or it may be used in a more direct fashion to compute physical quantities, such as stiffness and mass matrices, that can be interrogated to understand the structure's behavior. When the number of measured DOFs is larger than the number of identified modes, direct procedures such as in Ref. 2 for computing a global stiffness matrix directly from these limited data will fail. Thus, it is not always feasible to obtain the global stiffness matrix directly from modal test data. However, one can construct a rank-deficient flexibility matrix defined as

$$F_g = \Phi\Omega^{-2}\Phi^T \tag{1}$$

Our present challenge is to extract the substructural stiffness or substructural flexibility matrices from the preceding system-identified deficient global flexibility. Note that, in some cases, estimates of residual flexibility for each input–output pairing may also be obtained from experiment.[16] These can be utilized to enrich the global flexibility matrix and, hence, improve the identification of substructural flexibility.

The theoretical basis for deriving the global flexibility from substructural flexibilities is known as the force method, e.g., see Argyris and Kelsey.[19] For determinate structures, the force method yields the global flexibility matrix, e.g., see Felippa,[20] as

$$F_g = B_0^T F_e B_0 \tag{2}$$

where F_g is the global flexibility, F_e are the node-to-node flexibility matrices, and B_0 is the load transformation matrix from the applied loads to the internal force for determinate structures.

If the structure is statically indeterminate, one must obtain the so-called redundant load transformation matrix B_1 and modify Eq. (2) accordingly:

$$F_g = B_0^T\left[F_e - F_e B_1^T\left(B_1^T F_e B_1\right)^{-1} B_1^T F_e\right]B_0 \tag{3}$$

where B_1 is the transformation matrix that relates internal forces in so-called redundant elements to the resultant internal force distribution in the remaining nonredundant elements. Basically, if one were to determine the node-to-node substructural flexibility matrix F_e from Eqs. (2) and (3), one must first construct the load transformation matrices B_0 and B_1. Hence, a key feature for the extraction of F_e depends on the choice of the load transformation matrices B_0 and B_1. However, the difficulty in their unique determinations was a decisive reason in favoring the matrix stiffness method, which is now known as the finite element method. Note that a majority of real structures are of indeterminate type. Therefore, for continuum structures such as plates and shells, the node-to-node substructural flexibility F_e is difficult to define uniquely (although the resultant global flexibility is unique), which can lead to complexities in interpreting the extracted results. In addition, from a computational viewpoint, generalized inverses of B_0 and B_1 and their null-space bases that are required for extracting F_e present computational challenges.

The preceding observations motivated the present authors to employ a recently developed direct flexibility method[22] for the extraction of element-by-element substructural flexibility matrices from the measured frequencies and mode shapes.

III. Element-by-Element Substructural Flexibility

Consider the displacement-based finite element structural equilibrium equation given by

$$L^T K^{(s)} L u_g = f_g, \qquad K^{(s)} = \begin{bmatrix} K^{(1)} & & & \\ & K^{(2)} & & \\ & & \ddots & \\ & & & K^{(n_s)} \end{bmatrix} \tag{4}$$

where $\{L(n_s, n), n_s \geq n\}$ is the assembly matrix operator, $K^{(s)}$ is a block-diagonal matrix composed of the element-by-element stiffness matrices, u_g is the global nodal displacement vector, and f_g is the global external force vector, respectively.

Thus, if we express

$$L^T K^{(s)} L = K_g \tag{5}$$

then our objective will be accomplished if we obtain $K^{(s)}$ or its generalized inverse $F = K^{(s)+}$ from the expression (5), which is a special inverse problem. To this end, what we are about to employ is adapted from the so-called algebraically partitioned solution procedure for parallel computations of large-scale structural problems and its theoretical basis presented in terms of a direct flexibility method.[22] The essential idea of this algebraic partitioning is to decompose a global structure into a set of element-by-element substructures. This partitioning gives rise to two interface quantities: the Lagrange multipliers to account for the substructural interface forces and the rigid-body displacements for floating substructures. Hence, the solution of the substructural flexibility, viz., a generalized inverse of $K^{(s)}$, is in turn obtained by solving the two interface quantities.

To begin, we introduce the substructural displacement vector $\boldsymbol{d}$ and the substructural internal force p in terms of the global displacement vector u_g and the elemental stiffness matrix $K^{(s)}$, respectively,

$$\boldsymbol{d} = L u_g, \qquad \boldsymbol{p} = K^{(s)}\boldsymbol{d} = K^{(s)} L u_g \tag{6}$$

We now present a formulation for the derivation of elemental flexibility matrices in a step-by-step manner.

Step 1: Partitioning of the Global Equation into Substructural Equations

This step simply involves the algebraic decomposition of L^T, that is, the solution of

$$L^T \boldsymbol{p} = f_g \tag{7}$$

to yield

$$p = (L^T)^+ f_g - N\lambda$$
$$= f - N\lambda \quad (8)$$

where $f = Gf_g$ and where G is a generalized inverse of L^T, N is a null space basis of L^T, and the Lagrange multipliers λ are the complementary contributions to the solution of p due to algebraic partitioning. In physical terms, λ represents the interface forces along the substructural boundaries.

From the physical point of view, the null-space matrix N is the displacement compatibility operator that satisfies the following condition:

$$N^T d = 0, \quad \text{where} \quad d = Lu_g \quad (9)$$

Examples given in Sec. V offer how to construct the interface displacement compatibility operator N. A detailed algorithmic description of constructing N from the assembly matrix L is given in Ref. 22.

Step 2: Solution of Element-by-Element Displacement d

Using a pseudoinverse of the substructural stiffness $K^{(s)}$, one can solve for the substructural displacement vector from Eq. (6) as

$$d = K^{(s)+} p - Rd_r \quad (10)$$

where R is the orthonormalized null space basis for $K^{(s)}$, which are equivalently the orthonormalized (not mass-normalized) rigid-body modal vectors, and d_r is the substructural rigid-body displacement vector to be determined. Because $K^{(s)}$ is a stiffness matrix, its generalized inverse is a flexibility matrix that can be denoted by F and has the same domain-by-domain block diagonal form as $K^{(s)}$ [see Eq. (4)]. By the use of a spectral decomposition of $K^{(s)}$, viz.,

$$K^{(s)} = \Psi\Lambda\Psi^T, \qquad R^T\Psi = 0$$
$$\Psi\Psi^T + RR^T = I, \qquad \Psi^T\Psi = I \quad (11)$$

with Ψ as the orthonormal basis for $K^{(s)}$, we note that

$$[K^{(s)} + RR^T]^{-1} = \left([\Psi \quad R] \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^T \\ R^T \end{bmatrix} \right)^{-1}$$
$$= [\Psi \quad R] \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^T \\ R^T \end{bmatrix}$$
$$= \Psi\Lambda^{-1}\Psi^T + RR^T$$
$$= K^{(s)+} + RR^T$$
$$= F + RR^T \quad (12)$$

Therefore, we can compute F from $K^{(s)}$ using R as

$$F = [K^{(s)} + RR^T]^{-1} - RR^T \quad (13)$$

and rewrite Eq. (10) as

$$d = Fp - Rd_r \quad (14)$$

Note also that R satisfies the substructural static force equilibrium condition

$$R^T p = 0 \quad (15)$$

Furthermore, F possesses the complete deformation basis of $K^{(s)}$ with the same null space of $K^{(s)}$. This is in contrast to the elemental flexibility F_e in Eqs. (2) and (3), which is a nonsingular quantity based on user-defined constraints (and thus is not uniquely defined). We will call a generalized inverse of $K^{(s)}$ that satisfies the described property a statically complete subdomain flexibility. This property plays an important role not only in the computation of λ but also for computing d for a given external force.

By the substitution of p from Eq. (8) into Eq. (10), one obtains the substructural displacement given by

$$d = Lu_g = F\{f - N\lambda\} - Rd_r \quad (16)$$

Step 3: Global Displacement u_g from the Substructural Displacement d

The solution vector of the global system u_g can be obtained by a least-squares projection of the substructural-level solution d. This is accomplished from Eq. (16) as

$$u_g = G^T d, \qquad G = L(L^T L)^{-1}$$
$$= G^T F(f - N\lambda) - G^T Rd_r \quad (17)$$

Thus, the solution of the global problem is reduced to the solution of two variables, λ and d_r. This is addressed in the following step.

Step 4: Solution of λ and d_r

The three solution steps outlined so far can be brought together to form a coupled difference equation as follows. First, we impose the substructural static force equilibrium condition Eq. (15) to the substructural reaction force vector Eq. (8) to yield

$$R^T(f - N\lambda) = 0 \quad (18)$$

Second, we apply the elemental displacement compatibility condition Eq. (9–16) to obtain

$$N^T\{F(f - N\lambda) - Rd_r\} = 0 \quad (19)$$

The preceding two equations can be rearranged to form a coupled equation as

$$\begin{bmatrix} F_N & R_N \\ R_N^T & 0 \end{bmatrix} \begin{Bmatrix} \lambda \\ d_r \end{Bmatrix} = \begin{Bmatrix} N^T Ff \\ R^T f \end{Bmatrix} \quad (20)$$

where

$$F_N = N^T FN, \qquad R_N = N^T R \quad (21)$$

Step 5: Global Flexibility F_g from Elemental Flexibility Matrices F

Let us solve for λ from Eq. (20),

$$\lambda = F_N^{-1}(N^T Ff - R_N d_r) \quad (22)$$

Now substitute Eq. (22) into Eq. (21) to obtain d_r as

$$d_r = [K_R]^{-1}(R_N^T F_N^{-1} N^T F - R^T)f, \qquad K_R = R_N^T F_N^{-1} R_N \quad (23)$$

Finally, λ is obtained from Eqs. (22) and (23) as

$$\lambda = F_N^{-1} N^T Ff - F_N^{-1} R_N [K_R]^{-1}(R_N^T F_N^{-1} N^T F - R^T)f \quad (24)$$

Substituting λ and d_r into the global displacement equation (17), one finds that the global flexibility F_g is related to the elemental flexibility F according to

$$u_g = F_g f_g, \qquad F_g = G^T(F - FA - A^T F - FMF + F_R)G$$
$$G = L(L^T L)^{-1}, \qquad A = K_N F_R, \qquad M = K_N - K_N F_R K_N$$
$$K_N = NF_N^{-1} N_T, \qquad F_R = R\left[R^T K_N R\right]^{-1} R^T \quad (25)$$

Thus, Eq. (25) effectively assembles elemental or substructural flexibilities into the global flexibility. In contrast to the classical force method, e.g., see Argyris and Kelsey[19] and Felippa,[20] the present global flexibility given by Eq. (25) does not require any modeler-dependent assembly equations such as B_1 needed in the classical force method. In particular, with the substructural connectivity matrix L together with the elemental rigid-body modes R, the construction of the global flexibility is straightforward.

Note, however, that the present purpose is to extract the elemental flexibility or elemental stiffness matrices based on the experimentally determined global flexibility matrix F_g. This will be addressed in the next section.

IV. Extraction of Element-by-Element Flexibility from Measured Global Flexibility

To extract the element-by-element substructural flexibility from the experimentally determined global flexibility F_g, the present approach calls for two stages. First, we seek an iterated substructural flexibility from the formula derived in Eq. (25). It turns out that the substructural flexibility matrices, although they are energywise converged, contain deformation mode shapes that are in general not kinematically admissible. Hence, the unwanted modes need to be projected out. We now present these two steps.

Step 1: Iterative Solution of F

Formula (25) derived in the preceding section, relating the substructural free–free flexibility F to the global flexibility F_g, can be used to obtain F via iterations as follows. First, we rewrite Eq. (25) as

$$LF_gL^T = F - FA - A^TF - FMF + F_R \tag{26}$$

and obtain from the experimentally determined global flexibility an initial estimate of F^0 by taking its block diagonal matrices as

$$F^0(j_s{:}j_s{+}m_s,\ j_s{:}j_s{+}m_s) = LF_gL^T(j_s{:}j_s{+}m_s,\ j_s{:}j_s{+}m_s) \tag{27}$$

where the indices j_s and m_s are the location indicator and the size of the sth element flexibility matrix.

Second, iterate on F using the following formula:

$$F^{k+1} - F^kA^k - A^{k^T}F^k - F^kM^kF^k + F_R^k = LF_gL^T$$
$$= L\left(\Phi\Omega^{-2}\Phi^T + F_g^{\text{residual}}\right)L^T \tag{28}$$

where we also require that F^{k+1} for use in the next iteration should again retain only its block-diagonal entries.

Step 2: Kinematically Admissible Substructural Flexibility

The preceding iterated substructural flexibility F matrices, although energywise converged, often possess modes that are not kinematically compatible. The most fundamental property that each floating substructure must possess is that the deformation modes of each substructure must be orthogonal to its rigid-body modes. If, in addition to the rigid-body modes, a specific set of substructural deformation modes are predetermined, the remainder of the deformation modes must also be orthogonal to the predetermined deformation modes. Because one does not generally know a priori what substructural deformation modes must be present, the most one can usually do is to orthogonalize the iterated flexibility matrices (which are spanned by the substructural deformation modes) with respect to the substructural rigid-body modes. This can be carried out by the following projection for each substructure:

$$F = P_RF^kP_R, \qquad P_R = I - RR^T \tag{29}$$

where F is designated as a kinematically admissible substructural flexibility matrix.

Once the element-by-element substructural flexibility matrices F are obtained, the corresponding stiffness matrices can be obtained by

$$K^{(s)} = P_R(F + RR^T)^{-1}P_R \tag{30}$$

As an example, for a free–free planar, i.e., motion restricted to a plane, beam, the elemental stiffness matrix is given by

$$K^{(s)} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{31}$$

The rigid-body modes are given by

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -ls \\ 0 & 2s \\ 1 & ls \\ 0 & 2s \end{bmatrix}, \qquad s = \sqrt{4 + l^2} \tag{32}$$

The kinematically admissible flexibility is obtained as

$$F = P_R\left[K^{(s)} + RR^T\right]^{-1}P_R \tag{33}$$

Observe the duality of Eqs. (30) and (33). It should be pointed out that this duality does not hold in the case of node-to-node flexibility matrices obtained by the classical force method.

Summary of Present Procedure

As noted in the preceding subsection, projections are applied to the iterated substructural flexibility matrices such that the flexibilities are orthogonal to the substructure rigid-body modes. It is recommended that these projections be applied at each iteration in the solution for F and, therefore, can be incorporated into the iteration formula Eq. (28). Furthermore, because the projection matrix P_R is orthogonal to F_R in Eqs. (25) and (28), the final iterative formula is significantly simplified, viz.,

$$F^{k+1} - F^kM^kF^k = P_R(LF_gL^T)P_R \tag{34}$$

where the global flexibility F_g is approximated by the measured modal flexibility matrix $\Phi\Omega^{-2}\Phi^T$, which, if possible, should also be enriched by estimates of the residual flexibility F_g^{residual} from the identified model, e.g., see Ref. 24.

Several nonlinear solution strategies may be applied to solve for F. Strategies include a Riccati-equation-based iteration, a homotopy method,[25] and the sequential quadratic programming methods,[26] among others. Based on our limited experience, a discrete homotopylike method has been implemented for the present solution.

V. Two Simple Examples: Determinate and Indeterminate Trusses

Before we demonstrate the procedure to realistic problems, we demonstrate the procedure using two simple examples: a three-DOF determinate spring–mass system and a three-DOF indeterminate spring–mass system as shown in Fig. 1.

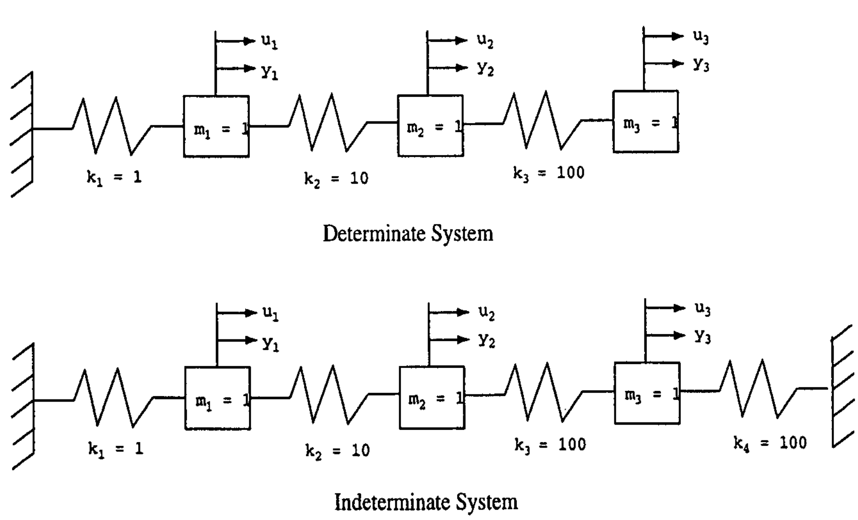

Fig. 1 Spring–mass system with three DOFs.

Example 1: Determinate Three-DOF Truss Problem

The global stiffness matrix K_g and the global flexibility matrix F_g are, respectively, given by

$$K_g = \begin{bmatrix} 11 & -10 & 0 \\ -10 & 110 & -100 \\ 0 & -100 & 100 \end{bmatrix}, \quad F_g = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.1 & 1.1 \\ 1 & 1.1 & 1.11 \end{bmatrix} \tag{35}$$

The element assembly operator L, the displacement compatibility matrix N, and the rigid-body modes R are obtained as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \tag{36}$$

because only elements 2 and 3, when disassembled, are in floating condition.

Iteration given by Eq. (27) yields the following iterated (marked by superscript k) elemental flexibility matrices:

$$\{F^1\}^k = 1.00, \qquad \{F^2\}^k = \begin{bmatrix} 1.00 & 1.00 \\ 1.00 & 1.10 \end{bmatrix}$$

$$\{F^3\}^k = \begin{bmatrix} 1.10 & 1.10 \\ 1.10 & 1.11 \end{bmatrix} \tag{37}$$

which, although energywise converged, do possess kinematically inadmissible modes except F^1. To project out the unwanted modes, we employ Eq. (29) to obtain

$$F^2 = 0.10 \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix}, \quad F^3 = 0.01 \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix} \tag{38}$$

Finally, by using Eq. (30) one obtains the elemental stiffness matrices as

$$K^1 = 1.0, \qquad K^2 = 10 \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix}$$

$$K^3 = 100 \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix} \tag{39}$$

which is the desired result.

Example 2: Indeterminate Three-DOF Truss Problem

For this case, the global stiffness matrix K_g is given by

$$K_g = \begin{bmatrix} 11 & -10 & 0 \\ -10 & 110 & -100 \\ 0 & -100 & 200 \end{bmatrix} \tag{40}$$

The element assembly operator L, the displacement compatibility matrix N, and the rigid-body mode matrix R are obtained as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \tag{41}$$

The iterated flexibility yields

$$\{F^1\}^k = 1.0000, \qquad \{F^2\}^k = \begin{bmatrix} 1.6610 \times 10^{-1} & 7.2818 \times 10^{-2} \\ 7.2818 \times 10^{-2} & 7.9539 \times 10^{-2} \end{bmatrix}$$

$$\{F^3\}^k = \begin{bmatrix} 3.6243 \times 10^{-2} & 1.7997 \times 10^{-2} \\ 1.7997 \times 10^{-2} & 9.7503 \times 10^{-3} \end{bmatrix}, \qquad \{F^4\}^k = 0.01 \tag{42}$$

Once again, although iterated values of $\{F^2\}^k$ and $\{F^3\}^k$ are in error, the use of projection transformation given by Eq. (29) recovers the correct elemental flexibility, which is then utilized via Eq. (30) to obtain the elemental stiffness matrices. The results are the same as obtained for the determinate case Eq. (39).

VI. Application to Complex Modeling: Engine Mount

Application of the deformation–displacement relation to the square box part of an engine-support ladder (Fig. 2) has been carried out to extract substructural flexibility. The ladder is modeled with eight planar beam elements, which include axial stiffness. In the numerical experiment, the global stiffness matrix is generated analytically. The global flexibility matrix is, thus, K_g^{-1} as the starting point. Following similar steps as employed in the preceding two examples, we obtained the kinematically admissible elemental flexibility from which the free–free elemental stiffness matrices are extracted.

Table 1 shows the convergence of the elemental eigenvalues of the horizontal midelement shown in Fig. 2. As can be seen in Table 1,

Table 1 Eigenvalues of horizontal midelement

Mode	Exact	Initial	Iterated
Bending 1	1.6666E+05	2.8992E+05	1.6672E+05
Bending 2	5.2000E+05	4.8771E+05	5.2003E+05
Axial	2.0000E+06	2.0011E+06	2.0000E+06

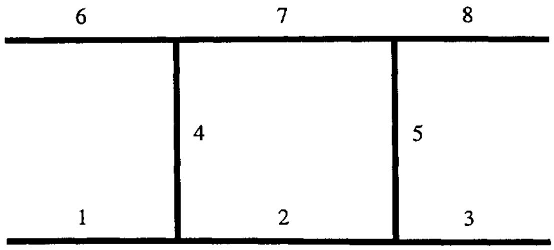

Fig. 2 Ladder modeled with eight planar beam elements.

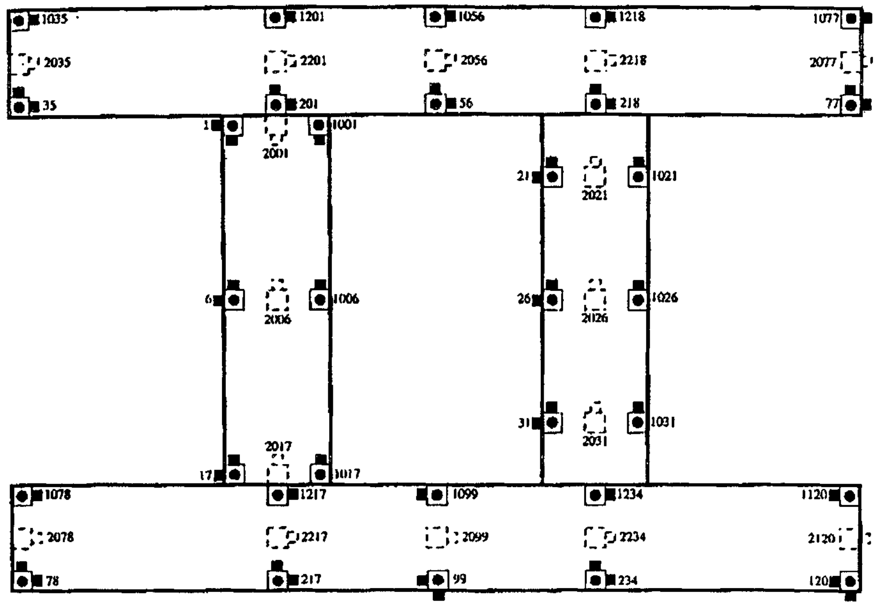

Fig. 3 Engine mount showing conceptual sensor configuration.

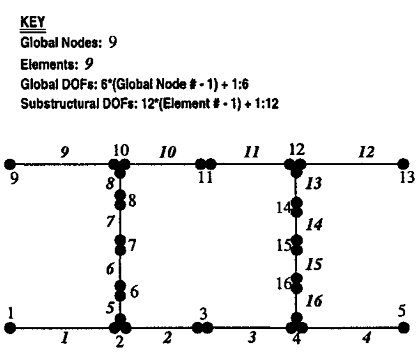

Fig. 4 Definition of the global and local DOFs based on sensors.

the initial errors of the two substructural bending eigenvalues are 74 and 6%, respectively. However, after iterations and filtering of unwanted modes, the extracted substructural stiffness matrix yields the two bending modes with accuracy in excess of four-digit accuracy.

We have also applied the procedure to a numerical simulation of partial damage in the ladder structure studied in the earlier example, with the objective of localizing the damage. To provide realism to the simulation, a high-fidelity 35,000-DOF plate element model of the welded tubular structure was utilized. The model is highly accurate and has been correlated to a modal test of a physical specimen using test modes up to 800 Hz (see Ref. 4). However, because we do not have experimental data from a damaged structure, we will utilize an incomplete set of frequencies and mode shapes as determined by our analytical model in a nominal and simulated damaged condition. The lowest 25 modes of the model representing the nominal undamaged structure were extracted, and global and substructural flexibility matrices based on 14 and 19 flexible modes were computed. To represent an example of partial damage to one of the joint welds, the element-to-element connections along the top and side of one of the four joints was released, to simulate a crack propagated along 50% of the joint circumference.

The sensor configuration assumed for this simulated experiment is shown in Fig. 3. The accelerometers are placed in sets of 6 at 16 different cross sections of the structure. This configuration allows us to define global and elemental node points with 6 DOFs per node, and to define elements or substructures connecting those nodes that are directly analogous to 12-DOF beam elements (although the element formulation is irrelevant to the procedure because we are just extracting the resultant flexibility). The definition of these global and local DOFs based on the sensor configuration is shown in Fig. 4.

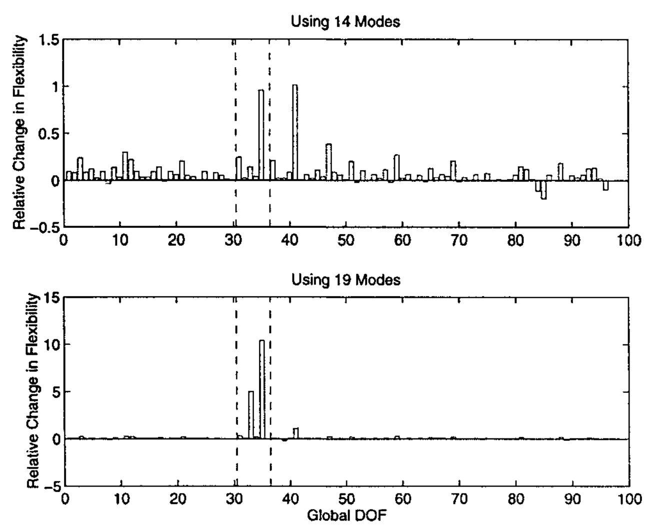

Fig. 5 Damage indicator based on global flexibility.

Figures 5 and 6 show the relative changes in the flexibility properties due to the induced damage based on using 14 or 19 modes to construct the flexibility. The dashed lines shown indicate the range of DOFs of the model (global or substructural) that are directly related to the damage location. Note that the global flexibility changes, using 14 modes to construct the flexibility matrix, indicate approximately the correct location, but other significant changes are seen across the structure. If 19 modes are used, the localization based on the global changes becomes much sharper, and the magnitude of the relative change is also more accurate. The damage indicator based on the substructural flexibilities, however, is very sharp for both the 14 mode and 19 mode cases, and in comparison with the global changes are more informative.

Thus, comparing the substructural-based vs global-based damage indications, we observe the distinct advantage of the present procedure. In passing, it should be noted that, by increasing the number of identified modes, one should eventually identify the locations based on either the global DOF or the local DOF changes. This may, however, not be feasible. On the other hand, the present substructural flexibility extraction procedure captures the relative substructure-by-substructure changes much more sharply than is typical by methods based on global changes.

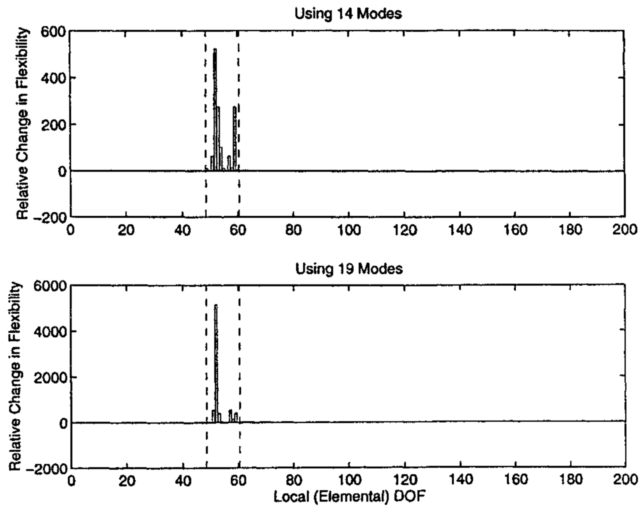

Fig. 6 Damage indicator based on substructural flexibility.

VII. Discussion and Conclusions

A method is presented for extracting the element-by-element substructural flexibility matrices from the measured structural frequencies and mode shapes. The present method utilizes an element-by-element direct flexibility method that assembles the global flexibility matrix from the free–free substructural flexibility matrices. Specifically, the present method consists of the following attributes:

1) Unlike the classical force method,[19] the method of assembling the global flexibility matrix does not depend on the ways the load paths are determined for redundant joints for statically indeterminate structures. The substructural rigid-body modes and the substructural connectivity topology uniquely determine the global flexibility matrix.

2)Because the iterated substructural flexibility matrices often embody the so-called kinematically inadmissible modes, a projection operator is introduced that filters out modes that are not orthogonal to the rigid modes. The remaining deformation modes consist of a consistent set of deformation modes plus the rigid-body modes. The resulting filtered flexibility matrix, termed herein kinematically admissible flexibility matrix, is used to obtain the substructural stiffness matrix.

3) For determinate structures, the method yields the substructural flexibility without iteration. For indeterminate structures, iterations and the use of the kinematically admissible flexibility matrix lead to the desired converged flexibility matrices (see the example problem in Sec. V and Table 1 of Sec. VI).

4) The present procedure is applied to the simulation of an engine mount beam with known joint damages. It is shown that the location of damages is more sharply identified by the present localized flexibility changes than is possible by the changes in the global flexibility matrix.

In addition, though not elaborated herein, the method has been applied to obtain an initial set of structural parameters for detailed model updating procedures. The preliminary experience indicates that the initially estimated parameters given by the method not only accelerate the parameter optimization computations, but more importantly enhance the feasible parameter ranges.

Acknowledgments

The work of the first author was supported by the U.S. Department of Energy under Contract DE-AC04-94AL85000. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the U.S. Department of Energy. It is a pleasure to acknowledge support for the second author by NASA Langley Research Center (Grant NAG1-1700) and Sandia National Laboratories (Contract AP-1461). We thank W. K. Belvin of NASA Langley Research Center and D. Martinez of Sandia for their enthusiasm and interest during the course of the research reported herein.

References

[1]Alvin, K. F., and Park, K. C., "Second-Order Structural Identification Procedure via State-Space-Based System Identification," *AIAA Journal*, Vol. 32, No. 2, 1994, pp. 397–406.

[2]Alvin, K. F., Peterson, L. D., and Park, K. C., "Method for Determining Minimal-Order Mass and Stiffness Matrices from Modal Test Data," *AIAA Journal*, Vol. 33, No. 1, 1995, pp. 128–135.

[3]Farhat, C., and Hemez, F. M., "Updating Finite Element Dynamic Models Using an Element-by-Element Sensitivity Methodology," *AIAA Journal*, Vol. 31, No. 9, 1993, pp. 1702–1711.

[4]Red-Horse, J. R., and Alvin, K. F., "Finite Element Modeling, Reconciliation, and Evaluation of Predictive Accuracy: A Case Study," *Proceedings of the AIAA/ASME 1996 Adaptive Structures Forum*, AIAA, Reston, VA, 1996, pp. 263–269.

[5]Doebling, S. W., Hemez, F. M., Barlow, M. S., Peterson, L. D., and Farhat, C., "Damage Detection in a Suspended Scale Model Truss via Modal Update," *Proceedings of the 11th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1993, pp. 1083–1094.

[6]Kaouk, M., and Zimmerman, D. C., "Structural Damage Assessment Using a Generalized Minimum Rank Perturbation Theory," *AIAA Journal*, Vol. 32, No. 4, 1994, pp. 836–842.

[7]Kim, H. M., and Bartkowicz, T. J., "An Experimental Study for Damage Detection Using a Hexagonal Truss Structure," *Proceedings of the AIAA/ASME 36th Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1995, pp. 3347–3356.

[8]Lim, T. W., and Kashangaki, T. A.-L., "Structural Damage Detection of Space Truss Structures Using Best Achievable Eigenvectors," *AIAA Journal*, Vol. 32, No. 5, 1994, pp. 1049–1057.

[9]Smith, S. W., and McGowan, P. E., "Locating Damaged Members in a Truss Structure Using Modal Data: A Demonstration Experiment," NASA TM-101595, April 1989.

[10]Zimmerman, D. C., and Kaouk, M., "Structural Damage Detection Using a Subspace Rotation Algorithm," *Proceedings of the AIAA/ASME 33rd Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1992, pp. 2341–2350.

[11]Zimmerman, D. C., and Kaouk, M., "Eigenstructure Assignment Approach for Structural Damage Detection," *AIAA Journal*, Vol. 30, No. 7, 1992, pp. 1848–1855.

[12]Robinson, N. A., Peterson, L. D., James, G. H., and Doebling, S. W., "Damage Detection in Aircraft Structures Using Dynamically Measured Static Flexibility Matrices," *Proceedings of the 14th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1996, pp. 857–865.

[13]Denoyer, K. K., and Peterson, L. D., "Method for Structural Model Update Dynamically Measured Static Flexibility Matrices," *AIAA Journal*, Vol. 35, No. 2, 1997, pp. 362–368.

[14]Denoyer, K. K., and Peterson, L. D., "Model Update Using Modal Contribution to Static Flexibility Error," *AIAA Journal*, Vol. 35, No. 11, 1997, pp.1739–1745.

[15]Doebling, S. W., Peterson, L. D., and Alvin, K. F., "Experimental Determination of Local Structural Stiffness by Disassembly of Measured Flexibility Matrices," *Journal of Vibration and Acoustics*, Vol. 120, No. 4, 1998, pp. 949–957.

[16]Doebling, S. W., "Measurement of Structural Flexibility Matrices for Experiments with Incomplete Reciprocity," Ph.D. Dissertation, Dept. of Aerospace Engineering Sciences, Rept. CU-CAS-95-10, Univ. of Colorado, Boulder, CO, April 1995.

[17]Hemez, F. M., "Advanced Tools for Updating Damped Finite Element Models Using Static, Modal and Flexibility Data," *Proceedings of the 14th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1996, pp. 511–517.

[18]Gordis, J. H., "On the Analytic Disassembly of Structural Matrices," *Proceedings of the 15th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1997, pp. 1292–1297.

[19]Argyris, J. H., and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworths, London, 1960.

[20]Felippa, C. A., "Parametric Unification of Matrix Structural Analysis: Classical Formulation and d-Connected Mixed Elements," *Finite Elements in Analysis and Design*, Vol. 21, No. 1, 1995, pp. 45–74.

[21]Farhat, C., and Roux, F.-X., "A Method of Finite Element Tearing and Interconnecting and its Parallel Solution Algorithm," *International Journal for Numerical Methods in Engineering*, Vol. 32, No. 6, 1991, pp. 1205–1227.

[22]Felippa, C. A., and Park, K. C., "A Direct Flexibility Method," *Computer Methods in Applied Mechanics and Engineering*, Vol. 149, No. 1–4, 1997, pp. 319–337.

[23]Park, K. C., Justino, M., and Felippa, C. A., "An Algebraically Partitioned FETI Method For Parallel Structural Analysis: Algorithm Description," *International Journal for Numerical Methods in Engineering*, Vol. 40, No. 15, 1997, pp. 2717–2737.

[24]Doebling, S. W., Peterson, L. D., and Alvin, K. F., "Estimation of Reciprocal Residual Flexibility from Experimental Modal Data," *AIAA Journal*, Vol. 34, No. 8, 1996, pp. 1678–1685.

[25]Richter, S. L., and Collins, E. G., Jr., "A Homotopy Algorithm for Reduced-Order Controller Design Using the Optimal Projection Equations," *Proceedings of the IEEE Conference on Decision and Control*, Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1989, pp. 506–511.

[26]Fletcher, R., *Practical Methods of Optimization*, 2nd ed., Wiley, New York, 1987, pp. 304–317.

A. Berman
Associate Editor